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Simultaneous estimations of temperature-dependent thermal conductivity and heat capacity

HAN-TAW CHEN†

Department of Mechanical Engineering, National Cheng Kung University, Tainan, Taiwan 701

and

JAE-YUH LIN

Chang Jung University, Tainan, Taiwan 711

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Abstract—A hybrid numerical algorithm of the Laplace transform technique and the control-volume method is proposed to simultaneously estimate the temperature-dependent thermal conductivity and heat capacity from temperature measurements inside the material. But, the functional forms of the thermal conductivity and heat capacity are unknown *a priori*. The whole domain is divided into several sub-layers and then the thermal properties in each sub-layer are assumed to be linear functional forms of temperature before performing the inverse calculation. The accuracy and efficiency of the predicted results can be evidenced from various illustrated cases using simulated exact and inexact temperature measurements obtained within the medium. Results show that good estimations on the thermal properties can be obtained from the knowledge of the transient temperature recordings only at two selected locations. The advantage of the present method is that the relation between the thermal properties and temperature can be determined for various types of boundary conditions even though the early temperature data cannot be obtained.

INTRODUCTION

Quantitative understanding of the heat transfer processes occurring in the industrial applications requires accurate knowledge of the thermal properties of the material. In practical situations measurements are often made of temperature, displacement, etc. Afterward, these measurements are fitted and then physical quantities or surface conditions may be estimated by using these curve-fitted measurements. Such problems are called inverse problems and have become an interesting subject recently. To data, various methods have been developed for the analysis of the inverse heat conduction problems involving the estimation of thermophysical properties from measured temperatures inside the material [1–13]. Beck and Al-Araji [1] applied the simple transient method to estimate the specific heat, thermal diffusivity and contact conductance. In this work the thermal conductivity is assumed to be constant or a linear function of temperature. Fukai *et al.* [2] used the nonlinear least-squares method to simultaneously estimate the thermal conductivity and specific heat. Flach and Özisik [3] and Huang and Özisik [5], respectively, employed the least-squares method and a direct integration approach to estimate spatially varying thermal conductivity and heat capacity. Afterward, Huang and Özisik [6] again applied a direct integration approach to estimate temperature-dependent thermal conductivity and heat capacity. Tervola [4] applied the finite element method in conjunction with the Davidon–Fletcher–Powell method to determine temperature-dependent thermal conductivity. It can be found that nine thermocouples were required in his work. His method also had some disadvantages that it must take quite a lot of computer time to obtain the estimation of the thermal conductivity, and the functional form of the thermal conductivity in the whole temperature interval must be given. In addition, his method was also sensitive to the selected locations of the thermocouples. This is unrealistic in real applications because the exact functional form of thermophysical properties is difficult to be defined before making the estimation. It can be found from these previous works [3–6] that good initial guesses were required to obtain the accurate predictions. Recently, Huang and co-workers [9, 10] utilized the conjugate gradient method to estimate the temperature-dependent thermal conductivity $k(T)$ with $\Delta x = 0.1$ and $\Delta t = 0.1$ and to simultaneously estimate the heat capacity $C(T)$ and $k(T)$ with $\Delta x = 0.1$ and $\Delta t = 0.02$. It can be seen from Ref. [9] that at least two ther-

† Author to whom correspondence should be addressed.

NOMENCLATURE

C	heat capacity per unit volume
C_0	reference heat capacity per unit volume
C_{err}	average of the absolute values of the relative errors for C
k	thermal conductivity
k_0	reference thermal conductivity
k_{err}	average of the absolute values of the relative errors for k
l	distance between two nodes
L	thickness of the slab
m	number of sub-layers
n	total number of nodes
p	Laplace transform parameter
q	surface heat flux
r	number of temperature sensors
s	number of time readings
t	time
T	temperature
$\tilde{T}$	transformed temperature
$\bar{T}$	initially guessed temperature or previously iterated temperature
T_{cal}	calculated temperature
T_{cur}	curve-fitted temperature
T_{exa}	exact temperature
T_{in}	initial temperature
T_{mea}	measured temperature
x	space coordinate
z	total number of estimated coefficients.

Greek symbols

β	coefficient for the thermal conductivity
ε	random variable
σ	standard deviation
γ	coefficient for the heat capacity.

mocouples were used in the experimental apparatus for estimating $k(T)$. The advantage of these two works [9, 10] was that no prior information was used for the functional forms of the unknown thermal conductivity and heat capacity in inverse calculations. However, their disadvantage [9, 10] was that the boundary conditions must be subjected to the prescribed constant heat flux and two temperature measurements must be recorded at the locations near the boundaries of the tested material. Lam and Yeung [11] and Yeung and Lam [12] applied a second-order finite difference approximation and two finite difference procedures to estimate the thermal conductivity in a one-dimensional heat conduction domain. The key feature of these approaches was that *a priori* knowledge of the functional form for the unknown thermal conductivity was not required in inverse calculations.

Chen *et al.* [13] even used the hybrid application of the Laplace transform technique and the finite-difference method to estimate the temperature-dependent thermal conductivity using measured nodal temperatures inside the material at any specified time. However, at least five thermocouples must be inserted into the tested material in order to obtain a more accurate estimation of $k(T)$. As stated by Tervola [4], it is better to seek the thermal properties in the whole temperature interval provided that only a few thermocouples are required to measure temperature. Thus, the present study applies the hybrid application of the Laplace transform technique and the control-volume method to simultaneously estimate the unknown thermal conductivity and heat capacity of a homogeneous material from temperature measurements. In performing the numerical simulation, the tested material is divided into several sub-layers. Thermophysical properties in each sub-layer are assumed to be a linear function of temperature. The initially guessed values of the thermal properties for each sub-layer can arbitrarily be given before performing the inverse calculation. The computational procedure for the estimation of the thermal conductivity and heat capacity is performed repeatedly until the sum of the squares of the deviations between the calculated and measured temperatures is minimum.

To evidence the accuracy of the present numerical algorithm in simultaneously estimating the temperature-dependent thermal conductivity and heat capacity per unit volume from temperature measurements, an example with temperature-dependent thermal properties is illustrated. In scientific and engineering experiments, the measurement of temperature is, in general, somewhat inaccurate. This may be due to human error, but more often, it is due to inherent limitations in the equipment being used to make the measurements. In the inverse heat conduction problem (IHCP), slight inaccuracies in the measured interior temperatures can affect the accuracy of estimated thermal properties. Thus, the effect of measurement errors on the estimation of the thermal conductivity and heat capacity will be investigated in the present analysis.

MATHEMATICAL FORMULATION

Consider a one-dimensional homogeneous slab with thickness L which is initially at temperature T_{in}. The thermal conductivity $k(T)$ and heat capacity per unit volume $C(T)$ are unknown and will be determined. Various types of boundary conditions may be

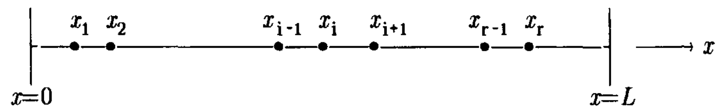

Fig. 1. Thermocouple arrangement for r temperature measurements.

involved in the present study. However, we select the problem that the boundary surface at $x = 0$ is subjected to a prescribed constant heat flux q and the other boundary surface at $x = L$ is kept insulated to illustrate the efficiency and accuracy of the present numerical algorithm. The governing differential equation of this problem can be expressed as

$$C(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] \quad \text{for } 0 < x < L, t > 0 \tag{1}$$

with boundary conditions in the following forms:

$$-k\frac{\partial T}{\partial x} = q \quad \text{at } x = 0 \tag{2}$$

$$\frac{\partial T}{\partial x} = 0 \quad \text{at } x = L \tag{3}$$

and the initial condition

$$T(x, 0) = T_{\text{in}}. \tag{4}$$

For the direct heat conduction problem, the temperature field in the slab as a function of space and time can be determined provided that all thermal properties of the slab are given. Conversely, unknown thermal properties need to be estimated unless additional information on temperature in the slab is given. To obtain the additional information, the temperature histories at some locations are usually measured in the slab. It is assumed that r thermocouples are used to record the temperature information at some selected locations, as shown in Fig. 1. The temperature histories taken from thermocouples are denoted by $T_{\text{mea}}(x_i, t_j)$, $i = 1, \ldots, r$ and $j = 1, \ldots, s$, where s denotes the number of the time readings.

The measured temperature data, T_{mea}, used in the present inverse analysis of the thermal conductivity and heat capacity can be determined with respect to the exact temperature solution of the direct heat conduction problem with the given thermal conductivity and heat capacity per unit volume, T_{exa}. However, owing to experimental uncertainty, T_{mea} should contain the measurement error. Thus, T_{exa} should be modified by adding small random errors to simulate experimental measurements. On the other hand, T_{mea} can be expressed as

$$T_{\text{mea}} = T_{\text{exa}} + \varepsilon\sigma \tag{5}$$

where the product of $\varepsilon\sigma$ represents the temperature measurement error and is assumed to be within $-0.05T_{\text{exa}}$ to $0.05T_{\text{exa}}$ in the present study. ε is a random variable that can be generated by subroutine DRNNOR of the IMSL [14] and lies in the range from -2.576 to 2.576 for normally distributed errors with zero mean and 99% confidence bounds. σ is the standard deviation of the temperature measurements with respect to the exact temperature data and is defined as

$$\sigma = \left\{\left[\sum_{j=1}^{s}\sum_{i=1}^{r}(T_{\text{mea}}(x_i, t_j) - T_{\text{exa}}(x_i, t_j))^2\right] \Big/ (r \times s)\right\}^{1/2}. \tag{6}$$

In real industrial applications, the actual measured temperature profiles often exhibit random oscillations owing to measurement errors. Based on the least-squares method, a polynomial function is used to fit the measured temperature data [15].

NUMERICAL ALGORITHM

In the present study, the slab is first divided into several sub-layers and then the thermal conductivity $k_i(T)$ and heat capacity $C_i(T)$ are assumed to vary linearly with temperature in each sub-layer. On the other hand, the functional form of $k_i(T)$ and $C_i(T)$ in each sub-layer can be assumed to be

$$k_i(T) = k_{0i}(1+\beta_i T) \quad \text{and} \quad C_i(T) = C_{0i}(1+\gamma_i T)$$
$$\text{for } i\text{th layer}, \quad i = 1, 2, \ldots, m \tag{7}$$

where m is the number of layers. k_0 and C_0 are the referenced thermal conductivity and heat capacity per unit volume, respectively. β_i and γ_i are coefficients corresponding to thermal conductivity and heat capacity per unit volume. Obviously, the requirement that the thermal properties at the interface between two adjacent layers are matched should be satisfied, i.e., $k_e = k_{e+1}$ and $C_e = C_{e+1}$. Thus, k_{0e} and C_{0e}, $e = 2, 3, \ldots, m$ can be obtained as

$$k_{0_e} = k_{0_{e-1}}\frac{1+\beta_{e-1}U_e}{1+\beta_e U_e} \text{ and } C_{0_e} = C_{0_{e-1}}\frac{1+\gamma_{e-1}U_e}{1+\gamma_e U_e},$$
$$e = 2, 3, \ldots, m \tag{8}$$

where U_e denotes the interface temperature between the eth layer and the $(e+1)$th layer. It is observed from equation (8) that $2(m+1)$ unknown variables, $\{k_{0_1}, \beta_1, \ldots, \beta_m, C_{0_1}, \gamma_1, \ldots, \gamma_m\}$, will be estimated and can be expressed as

$$\{Y_i\}_{i=1}^{z} = \{k_{0_1}, \beta_1, \ldots, \beta_m, C_{0_1}, \gamma_1, \ldots, \gamma_m\} \tag{9}$$

where $z = 2(m+1)$.

Owing to the above assumptions, the hybrid application of the Laplace transform technique and the control-volume method proposed by Chen and Lin [16] can be used to solve the present problem. First, equations (1) and (2) are rewritten as

$$\frac{\partial^2 K}{\partial x^2} = \frac{\partial H}{\partial t} \quad \text{for } 0 < x < L, t > 0 \tag{10}$$

$$-\frac{\partial K}{\partial x} = q \quad \text{at } x = 0 \tag{11}$$

where K and H are defined as

$$K(T) = \int k(T)\,\mathrm{d}T \quad \text{and} \quad H(T) = \int C(T)\,\mathrm{d}T. \tag{12}$$

The Laplace transforms of equations (3), (10) and (11) with respect to t are

$$\frac{\mathrm{d}^2\tilde{K}}{\mathrm{d}x^2} - p\tilde{H} = -H(T_{\text{in}}) \quad \text{for } 0 < x < L \tag{13}$$

$$-\frac{\mathrm{d}\tilde{K}}{\mathrm{d}x} = \frac{q}{p} \quad \text{at} \quad x = 0 \tag{14}$$

and

$$-\frac{\mathrm{d}\tilde{T}}{\mathrm{d}x} = 0 \quad \text{at} \quad x = L \tag{15}$$

where $\tilde{K}$, $\tilde{H}$ and $\tilde{T}$, respectively, denote the Laplace transform of K, H and T. p is the Laplace transform parameter.

The linearization forms of the function $\tilde{K}$ and $\tilde{H}$ using the Taylor's series approximation are [16]

$$\tilde{K} = k(\bar{T})\tilde{T} + \frac{1}{p}[K(\bar{T}) - k(\bar{T})\bar{T}] \tag{16}$$

and

$$\tilde{H} = C(\bar{T})\tilde{T} + \frac{1}{p}[H(\bar{T}) - C(\bar{T})\bar{T}] \tag{17}$$

where $\bar{T}$ is the initially guessed temperature or previously iterated temperature.

Owing to the application of the control-volume method, the integration of equation (13) over a typical control volume $[x_j - l/2, x_j + l/2]$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tilde{K})_{x=x_j+l/2} - \frac{\mathrm{d}}{\mathrm{d}x}(\tilde{K})_{x=x_j-l/2} - p\int_{x_j-l/2}^{x_j+l/2} \tilde{H}\,\mathrm{d}x = -\int_{x_j-l/2}^{x_j+l/2} H(T_{\text{in}})\,\mathrm{d}x \tag{18}$$

where l is the distance between two nodes and is taken as uniform in the present study.

$\tilde{T}$ within the interval $[x_{j-1}, x_{j+1}]$ can be approximated in terms of the shape functions and unknowns transformed nodal temperatures as

$$\begin{aligned}\tilde{T}(x) &= \frac{x - x_{j-1}}{l}\tilde{T}_j + \frac{x_j - x}{l}\tilde{T}_{j-1} \quad \text{for } x \in [x_{j-1}, x_j] \\ &= \frac{x_{j+1} - x}{l}\tilde{T}_j + \frac{x - x_j}{l}\tilde{T}_{j+1} \quad \text{for } x \in [x_j, x_{j+1}].\end{aligned} \tag{19}$$

Similarly, $\bar{T}$ within the interval $[x_{j-1}, x_{j+1}]$ can also be approximated in the same form as equation (19).

Substituting equations (16), (17) and (19) in conjunction with equations (7) and (8) into equation (18) yields an algebraic equation for each interior node as

$$u_j\tilde{T}_{j-1} + v_j\tilde{T}_j + w_j\tilde{T}_{j+1} = g_j, \quad j = 2, 3, \ldots, n-1 \tag{20}$$

where n is the total number of nodes. The coefficients u_j, v_j, w_j and g_j for nodes within the eth layer are

$$u_j = k_{0e}(1 + \beta_e\bar{T}_{j-1}) - \frac{C_{0e}pl^2}{8}\left[1 + \frac{\gamma_e}{3}(2\bar{T}_j + \bar{T}_{j-1})\right] \tag{21a}$$

$$v_j = -2k_{0e}(1 + \beta_e\bar{T}_j) - \frac{C_{0e}pl^2}{8}\left[6 + \frac{\gamma_e}{3}(2\bar{T}_{j-1} + 14\bar{T}_j + 2\bar{T}_{j+1})\right] \tag{21b}$$

$$w_j = k_{0e}(1 + \beta_e\bar{T}_{j+1}) - \frac{C_{0e}pl^2}{8}\left[1 + \frac{\gamma_e}{3}(2\bar{T}_j + \bar{T}_{j+1})\right] \tag{21c}$$

and

$$g_j = \frac{k_{0e}\beta_e}{2p}(\bar{T}_{j-1}^2 - 2\bar{T}_j^2 + \bar{T}_{j+1}^2) - C_{0e}l^2(1 + \gamma_e T_{\text{in}})T_{\text{in}} - \frac{C_{0e}\gamma_e l^2}{48}(\bar{T}_{j-1}^2 + 4\bar{T}_{j-1}\bar{T}_j + 14\bar{T}_j^2 + 4\bar{T}_j\bar{T}_{j+1} + \bar{T}_{j+1}^2). \tag{21d}$$

The coefficients u_j, v_j, w_j and g_j for nodes at the interface between the eth layer and the $(e+1)$th layer are

$$u_j = k_{0e}(1 + \beta_e\bar{T}_{j-1}) - \frac{C_{0e}pl^2}{8}\left[1 + \frac{\gamma_e}{3}(2\bar{T}_j + \bar{T}_{j-1})\right] \tag{22a}$$

$$v_j = -k_{0e}(1 + \beta_e\bar{T}_j) - k_{0e+1}(1 + \beta_{e+1}\bar{T}_j) - \frac{C_{0e}pl^2}{8}\left[3 + \frac{\gamma_e}{3}(2\bar{T}_{j-1} + 7\bar{T}_j)\right] - \frac{C_{0e+1}pl^2}{8}\left[3 + \frac{\gamma_{e+1}}{3}(7\bar{T}_j + 2\bar{T}_{j+1})\right] \tag{22b}$$

$$w_j = k_{0e+1}(1 + \beta_{e+1}\bar{T}_{j+1}) - \frac{C_{0e+1}pl^2}{8}\left[1 + \frac{\gamma_{e+1}}{3}(2\bar{T}_j + \bar{T}_{j+1})\right] \tag{22c}$$

and

$$g_j = \frac{1}{2p}[k_{0e}\beta_e(\tilde{T}_{j-1}^2 - \tilde{T}_j^2) + k_{0e+1}\beta_{e+1}(\tilde{T}_{j+1}^2 - \tilde{T}_j^2)]$$

$$-\frac{l^2}{2}[C_{0e}(1+\gamma_e T_{in}) + C_{e+1}(1+\gamma_{e+1}T_{in})]T_{in}$$

$$-\frac{l^2}{48}[C_e\gamma_e(\tilde{T}_{j-1}^2 + 4\tilde{T}_{j-1}\tilde{T}_j + 7\tilde{T}_j^2)$$

$$+C_{e+1}\gamma_{e+1}(7\tilde{T}_j^2 + 4\tilde{T}_j\tilde{T}_{j+1} + \tilde{T}_{j+1}^2)]. \quad (22d)$$

The algebraic equations for nodes on the boundary surfaces at $x = 0$ and $x = L$ can also be obtained in the similar way as

$$v_1\tilde{T}_1 + w_1\tilde{T}_2 = g_1 \quad (23)$$

and

$$u_n\tilde{T}_{n-1} + v_n\tilde{T}_n = g_n \quad (24)$$

where

$$v_1 = -k_{01}(1+\beta_1\tilde{T}_1) - \frac{C_{01}pl^2}{8}\left[3 + \frac{\gamma_1}{3}(7\tilde{T}_1 + 2\tilde{T}_2)\right] \quad (25a)$$

$$w_1 = k_{01}(1+\beta_1\tilde{T}_2) - \frac{C_{01}pl^2}{8}\left[1 + \frac{\gamma_1}{3}(2\tilde{T}_1 + \tilde{T}_2)\right] \quad (25b)$$

$$g_1 = \frac{k_{01}\beta_1}{2p}(\tilde{T}_2^2 - \tilde{T}_1^2) - \frac{C_{01}l^2}{2}(1+\gamma_1 T_{in})T_{in}$$

$$-\frac{C_{01}\gamma_1 l^2}{48}(7\tilde{T}_1^2 + 4\tilde{T}_1\tilde{T}_2 + \tilde{T}_2^2) - \frac{lq}{p} \quad (25c)$$

and

$$u_n = k_{0m}(1+\beta_m\tilde{T}_{n-1}) - \frac{C_{0m}pl^2}{8}\left[1 + \frac{\gamma_m}{3}(\tilde{T}_{n-1} + 2\tilde{T}_n)\right] \quad (26a)$$

$$v_n = -k_{0m}(1+\beta_m\tilde{T}_n) - \frac{C_{0m}pl^2}{8}\left[3 + \frac{\gamma_m}{3}(2\tilde{T}_{n-1} + 7\tilde{T}_n)\right] \quad (26b)$$

$$g_n = \frac{k_{0m}\beta_m}{2p}(\tilde{T}_{n-1}^2 - \tilde{T}_n^2) - \frac{C_{0m}l^2}{2}(1+\gamma_m T_{in})T_{in}$$

$$-\frac{C_{0m}\gamma_m l^2}{48}(\tilde{T}_{n-1}^2 + 4\tilde{T}_{n-1}\tilde{T}_n + 7\tilde{T}_n^2). \quad (26c)$$

The rearrangement of equations (20), (23) and (24) can yield the following matrix equation.

$$[A]\{\tilde{T}\} = \{f\} \quad (27)$$

where [A] is an $n \times n$ band matrix containing the Laplace transform parameter p. $\{\tilde{T}\}$ is an $n \times 1$ matrix representing the unknown nodal temperatures in the transform domain and $\{f\}$ is an $n \times 1$ matrix representing the forcing terms. The calculated nodal temperatures can be obtained from equation (27) by using the Gaussian elimination algorithm and the numerical inversion of Laplace transform [17] provided that the estimated thermal conductivity and heat capacity are given.

In the present study, the least-squares minimization technique will be applied to minimize the sum of the squares of the deviations between the calculated and curve-fitted nodal temperatures. This implies that the function

$$\mathrm{F}(Y_1, Y_2, \ldots, Y_z) = \sum_{i=1}^{z}(T_{\mathrm{cal},i} - T_{\mathrm{cur},i})^2 \quad (28)$$

is to be minimized. $T_{\mathrm{cur},i}$, $i = 1, 2, \ldots, z$, is determined from the curve-fitted temperature profile. The estimated values of $\{Y_i\}_{i=1}^z$ are determined until $\mathrm{F}(Y_1, Y_2, \ldots, Y_n)$ is minimum. The computational procedures for estimating the unknown coefficients, $\{Y_i\}_{i=1}^z$, are described as follows.

The initial guesses of $\{Y_i\}_{i=1}^z$ are chosen arbitrarily. Afterward, the calculated temperature data corresponding to the specified time, $\{T_{\mathrm{cal},i}\}_{i=1}^z$, can be determined from equation (27). Differences between $T_{\mathrm{cur},i}$ and $T_{\mathrm{cal},i}$ are expressed as

$$e_i = T_{\mathrm{cal},i} - T_{\mathrm{cur},i}, \quad i = 1, 2, \ldots, z. \quad (29)$$

Corrections $\{d_i\}_{i=1}^z$ for $\{Y_i\}_{i=1}^z$ are introduced in order to minimize the values of e_i. Thus the updated values of the estimated coefficients Y_i can be written as

$$Y_i = Y_i^* + d_i\delta_{ij}, \quad i,j = 1, 2, \ldots, z \quad (30)$$

where Y_i^* denotes the initially guessed value or previously predicted value of the estimated coefficient. The symbol δ_{ij} is Kronecker delta and is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \quad (31)$$

The new calculated temperatures $\{T_{\mathrm{cal},ij}, j = 1, 2, \ldots, z\}_{i=1}^z$ with respect to Y_i given by equation (30) can also be determined from equation (27). Accordingly, differences between the new calculated temperatures and curve-fitted temperatures can be written as

$$e_{ij} = T_{\mathrm{cal},ij} - T_{\mathrm{cur},i}, \quad i,j = 1, 2, \ldots, z. \quad (32)$$

Differences between e_i and e_{ij} with respect to corrections $\{d_i\}_{i=1}^z$ are defined as

$$\omega_{ij} = (e_{ij} - e_i)/d_i, \quad i,j = 1, 2, \ldots, z. \quad (33)$$

The total differences between $T_{\mathrm{cal},ij}$ and $T_{\mathrm{cur},i}$ can be written as

$$\lambda_i = \sum_{j=1}^{z} e_{ij}, \quad j = 1, 2, \ldots, z. \quad (34)$$

The sum of λ_i^2 can be expressed as

$$\mathbf{E} = \sum_{i=1}^{z} \lambda_i^2. \quad (35)$$

To determine the desired corrections for the guessed thermal conductivity and heat capacity, minimizing $\mathbf{E}$ with respect to $\{d_i\}_{i=1}^{z}$ will be performed. Differentiating $\mathbf{E}$ with respect to each correction yields

$$\sum_{i=1}^{z} \sum_{j=1}^{z} (\omega_{ji}\omega_{jk}d_i) = -\sum_{j=1}^{z} \omega_{ik}e_i \quad k = 1, 2, \ldots, z. \quad (36)$$

Equation (36) is a set of z algebraic equations. New corrections $\{d_i\}_{i=1}^{z}$ can be obtained by solving equation (36). The above procedures are repeated until the differences between the calculated temperatures and the curve-fitted temperatures are all less than a tolerable value. In the present study, the tolerable value is chosen as 10^{-4}.

RESULTS AND DISCUSSION

To assess the accuracy of the present numerical algorithm in predicting the unknown thermal conductivity and heat capacity from the knowledge of temporal temperature recordings at selected locations, an example involving the prediction of temperature-dependent thermal properties will be illustrated. In the meantime, the effects of measurement errors, the number of thermocouples and time readings on the estimation of the thermal conductivity and heat capacity are also examined. For the convenience of numerical analysis, the thickness of the slab, L, and surface heat flux, q, are taken as unity. The initial temperature, T_{in}, is assumed to be zero. In all the test cases considered here, the initial guesses of estimated coefficients, $\{Y_i\}_{i=1}^{z}$, used to begin the iteration are taken as unity. First, assume that 11 temperature sensors with spacing $\Delta x = 0.1$ are inserted into the tested material to record the temperature data. For each temperature sensor, 21 temperature data are read from $t = 0.1$ to $t = 2.1$ with the time interval $\Delta t = 0.1$. This implies that 231 temperature data will be obtained.

The inverse problem that the exact thermal conductivity $k(T)$ is a combination of the exponential function and a second-order polynomial with temperature, and the exact heat capacity $C(T)$ is a second-order polynomial with temperature is illustrated. The functional forms of $k(T)$ and $C(T)$ are assumed as follows.

$$k(T) = 1 + T^2 + \exp(T) \quad \text{and} \quad C(T) = 1 + T + T^2. \quad (37)$$

Figures 2 and 3, respectively, show the estimated results of the thermal conductivity and heat capacity using 11 temperature sensors with spacing $\Delta x = 0.1$ for $\sigma = 0$ and $\sigma = 0.005$. Numerical results shown in these two figures are obtained under the condition that the slab is divided into 10 sub-layers with equal space. This implies that 22 unknown coefficients are to be determined. Thus we must take 22 measured temperature data recorded at any two specified time t_1 and t_2 to determine these unknown coefficients. For the convenience of numerical analysis, assume that the temperature information is obtained at $t_1 = 1.0$ and $t_2 = 2.0$. It can be seen from Figs. 2 and 3 that the estimated thermal properties are in good agreement with the exact solutions for the case of $\sigma = 0$ or for the assumption of no measurement error. However, the inaccuracy of the estimated thermal properties for the case of $\sigma = 0.005$ will occur owing to the measurement errors of the experiment. The average of the absolute values of the relative errors between the exact values and the estimated values for $k(T)$ and $C(T)$ is, respectively, defined as

$$k_{\text{err}} = \left[\sum_{j=1}^{s} \sum_{i=1}^{r} \left| \frac{k_{\text{ex}}(x_i, t_j) - k_{\text{es}}(x_i, t_j)}{k_{\text{ex}}(x_i, t_j)} \right| \right] \Big/ (r \times s) \times 100\% \quad (38a)$$

and

$$C_{\text{err}} = \left[\sum_{j=1}^{s} \sum_{i=1}^{r} \left| \frac{C_{\text{ex}}(x_i, t_j) - C_{\text{es}}(x_i, t_j)}{C_{\text{ex}}(x_i, t_j)} \right| \right] \Big/ (r \times s) \times 100\% \quad (38b)$$

where the subscripts es and ex denote the estimated and exact values, respectively.

The values of k_{err} and C_{err} for 11 temperature sensors are $k_{\text{err}} = 0.005\%$ and $C_{\text{err}} = 0.007\%$ for $\sigma = 0$ and $k_{\text{err}} = 0.974\%$ and $C_{\text{err}} = 1.61\%$ for $\sigma = 0.005$, respectively. For practical application, the variations of the estimated thermal conductivity $k_{\text{es}}(T)$ and estimated heat capacity per unit volume $C_{\text{es}}(T)$ with temperature shown in Figs. 2 and 3 are presented using the simple general expressions. These empirical correlations which represent all of the predicted results within 0.2% in the whole temperature interval are constructed. These correlations can be expressed as

$$k_{\text{es}}(T) = 1.35609 + 8.9148T - 20.4444T^2 + 22.7374T^3 - 7.80911T^4 \quad (39)$$

and

$$C_{\text{es}}(T) = -7.32923 + 44.0632T - 79.8718T^2 + 65.6378T^3 - 19.4672T^4. \quad (40)$$

To investigate the effect of the temperature data obtained from various combinations of two specified times, t_1 and t_2, on the estimation of thermal properties, the predicted results of the thermal conductivity and heat capacity per unit volume using 11 temperature sensors are shown in Table 1. No obvious difference between them is found. This implies that the effect of the selection of t_1 and t_2 on the estimation of thermal properties using the present numerical method is little. To further evidence the efficiency of

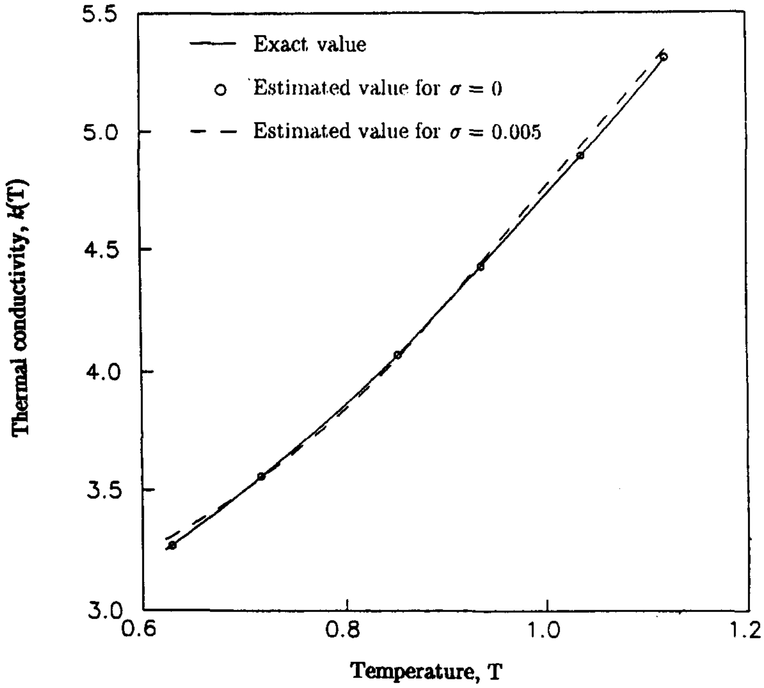

Fig. 2. Comparison of the exact and predicted values of $k(T)$.

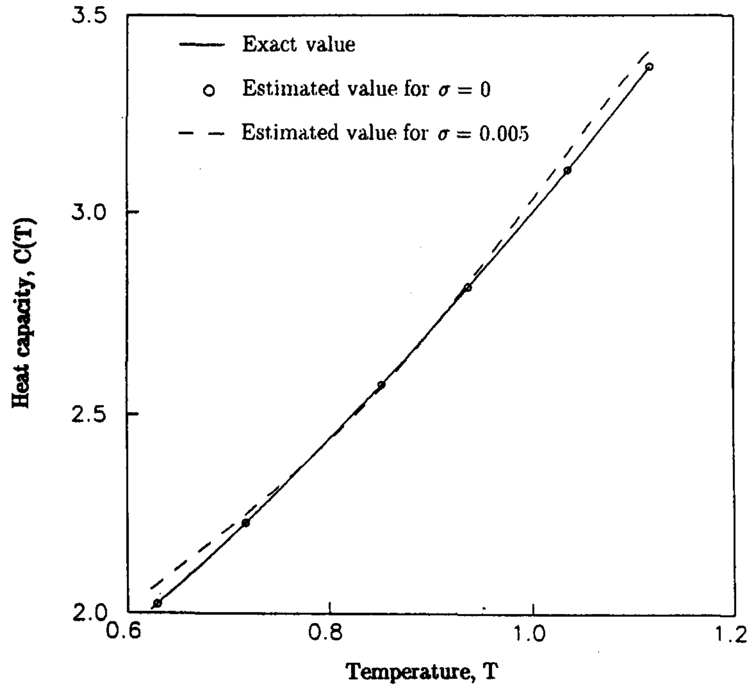

Fig. 3. Comparison of the exact and predicted values of $C(T)$.

the present numerical simulation for the thermal properties, the effect of temperature histories recorded at two selected locations, x_1 and x_2 from $t = 0.1$ to $t = 2.1$ with the time step $\Delta t = 0.2$ on the estimated results is also investigated, as shown in Table 2. Results show that the estimated results obtained from the present method are insensitive to the inserted locations of the thermocouples even for the case with the measurement error.

In all computations of this illustrated example,

Table 1. Effect of (t_1, t_2) on the estimations of $k(T)$ and $C(T)$

	$\sigma = 0$		$\sigma = 0.005$	
(t_1, t_2)	k_{err} (%)	C_{err} (%)	k_{err} (%)	C_{err} (%)
(0.1, 0.2)	0.043	0.050	1.001	1.771
(0.1, 0.5)	0.038	0.044	0.998	1.763
(0.1, 1.0)	0.005	0.007	0.987	1.610
(0.1, 2.1)	0.001	0.001	0.965	1.411
(0.5, 1.0)	0.006	0.009	0.976	1.611
(1.0, 2.1)	0.004	0.007	0.971	1.526
(2.0, 2.1)	0.043	0.050	1.000	1.732

Table 2. Effect of (x_1, x_2) on the estimations of $k(T)$ and $C(T)$

	$\sigma = 0$		$\sigma = 0.005$	
(x_1, x_2)	k_{err} (%)	C_{err} (%)	k_{err} (%)	C_{err} (%)
(0.0, 0.1)	0.082	0.090	1.230	1.785
(0.0, 0.5)	0.071	0.077	1.213	1.696
(0.0, 1.0)	0.055	0.060	1.191	1.547
(0.1, 0.2)	0.082	0.087	1.191	1.784
(0.1, 0.9)	0.058	0.063	1.196	1.578
(0.5, 0.6)	0.081	0.086	1.227	1.783
(0.5, 0.9)	0.071	0.078	1.211	1.696

seven iterations are required to obtain the predicted results of the thermal conductivity and heat capacity. The CPU time for a PC-586 computer with pentium-100 is about 40 s.

CONCLUSIONS

The present study proposes a numerical simulation involving the Laplace transform technique and the control-volume method in conjunction with the least-square method to simultaneously estimate the unknown thermal conductivity and unknown heat capacity per unit volume from temperature measurements inside the tested material. Owing to the application of the Laplace transform scheme, the thermal conductivity and heat capacity can be estimated simultaneously even though the early temperature data cannot be recorded. It is found from the illustrated example that the present numerical algorithm can give a good estimation of the thermal conductivity and heat capacity simultaneously even for the problem with temperature measurement errors. The advantages of the present numerical algorithm are not very sensitive to the choice of initial guesses, measurement locations and measurement times. It is worth mentioning that good estimations on the thermal conductivity and heat capacity can also be obtained only using two temperature sensors.

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REFERENCES

1. Beck, J. V. and Al-Araji, S., Investigation of a new simple transient method of thermal property measurement. *ASME Journal of Heat Transfer*, 1974, **96,** 59–64.
2. Fukai, J., Watanabe, M., Miura, T. and Ohtani, S., Simultaneous estimation of thermophysical properties by nonlinear least-squares. *Int. Chem. Engng*, 1987, **87,** 455–465.
3. Flach, G. P. and Özisik, M. N., Inverse heat conduction problem of simultaneously estimating spatially varying thermal conductivity and heat capacity per unit volume. *Numer. Heat Transfer*, 1989, **A16,** 249–266.
4. Tervola, P., A method to determine the thermal conductivity from measured temperature profiles. *International Journal of Heat and Mass Transfer*, 1989, **32,** 1425–1430.
5. Huang, C. H. and Özisik, M. N., A direct integration approach for simultaneously estimating spatially varying thermal conductivity and heat capacity. *International Journal of Heat and Fluid Flow*, 1990, **11,** 262–268.
6. Huang, C. H. and Özisik, M. N., Direct integration approach for simultaneously estimating temperature-dependent thermal conductivity and heat capacity. *Numer. Heat Transfer*, 1991, **A20,** 95–110.
7. Artyukhin, E. A., Iterative algorithms for estimating temperature-dependent thermophysical characteristics. *Inverse Problems in Engineering: Theory and Practice.* The first conference in a series on inverse problems in engineering, ed. N. Zabaras *et al.*, Florida, 1993, pp. 101–108.
8. Shai, I. and Haim, M., High thermal conductivity measurement using internal heat source method. *Proceedings of the 22nd International Conference on Thermal Conductivity*, ed. T. W. Tong, 1994, pp. 814–825.
9. Huang, C. H., Yan, J. J. and Chen, H. T., Function estimation in predicting temperature-dependent thermal conductivity without internal measurements. *J. Thermophys. Heat Transfer*, 1995, **9,** 667–673.
10. Huang, C. H. and Yan, J. Y., An inverse problem in simultaneously measuring temperature-dependent thermal conductivity and heat capacity. *International Journal of Heat and Mass Transfer*, 1995, **38,** 3433–3441.
11. Lam, T. T. and Yeung, W. K., Inverse determination of thermal conductivity for one-dimensional problems. *J. Thermophys. Heat Transfer*, 1995, **9,** 335–344.
12. Yeung, W. K. and Lam, T. T., Second-order finite difference approximation for inverse determination of thermal conductivity. *International Journal of Heat and Mass Transfer*, 1995, **39,** 3685–3693.
13. Chen, H. T., Lin, J. Y., Wu, C. H. and Huang, C. H., Numerical algorithm for estimating temperature-dependent thermal conductivity. *Numer. Heat Transfer*, 1996, **B29,** 509–522.
14. *IMSL Library Edition* 10.0, *User's Manual*: *Math, Library Version* 1.0, *IMSL*, Houston, TX, 1987.
15. Holman, J. P. and Gajda, W. J. Jr, *Experimental Methods for Engineers*, 5th edn. McGraw-Hill, New York, 1989, pp. 37–83.
16. Lin, J. Y. and Chen, H. T., Numerical analysis for phase-change problems with the mushy zone. *Numer. Heat Transfer*, 1995, **A24,** 163–177.
17. Honing, G. and Hirdes, U., A method for the numerical inversion of Laplace transforms. *J. Comput. Appl. Math.*, 1984, **10,** 113–132.